

Variation on a Theorem by Mues and Steinmetz

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Abstract

Let f be a meromorphic function. We suggest a generalization of f and its derivative f' sharing a nonzero value a IM that does not impose any a priori restrictions on the ramification of f . Then we discuss some results around the question whether the famous theorem on entire functions f that share two values IM with f' still holds for this weaker notion.

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1. Introduction

Two meromorphic functions f and g on a complex domain D are said to share the value $a \in \mathbb{C}$ if for all $z \in D$ we have $f(z) = a \Leftrightarrow g(z) = a$. Usually this is more loosely written as $f = a \Leftrightarrow g = a$. Often one also says more precisely that f and g are sharing the value a IM (ignoring multiplicities) to emphasize that one does not require that at such points the function f takes the value a with the same multiplicity as g .

Here we are interested in the case where the domain D is the whole complex plane \mathbb{C} , f is an entire function, and g is its derivative f' . The most famous theorem in this setting is the following result from 1979.

Theorem A. [MuSt, Satz 1] *Let f be a nonconstant entire function and let a and b be complex numbers with $a \neq b$. If f and f' share the values a and b IM, then $f \equiv f'$.*

See also [YaYi, Theorem 8.3] for a proof in English. A different proof for the case $ab \neq 0$ was also given in [Gu, Theorem 2].

Theorem A has been generalized in many ways. We will use a relatively new result which gives up one of the directions of the implication $f = b \Leftrightarrow f' = b$. Actually, it also provides a third different proof for the case $ab \neq 0$ of Theorem A.

Theorem B. [LüXuYi, Theorem 1.1] *Let a and b be two nonzero distinct complex numbers, and let $f(z)$ be a nonconstant entire function. If $f(z) = a \Leftrightarrow f'(z) = a$ and $f'(z) = b \Rightarrow f(z) = b$, then $f \equiv f'$.*

The functions $f = \frac{b}{4}z^2$ and $f = Ce^{\frac{a}{b}z} + a - C$ from [LiYi] show that the conditions $a \neq 0$ and $b \neq 0$ cannot be removed in Theorem B. Here C is any nonzero constant.

There is also a result that gives up the other direction of the implication. It provides an alternative proof for the case $ab = 0$ of Theorem A.

Theorem C. [LüXuYi, special case of Corollary 1.1] *Let b be a nonzero number, and let $f(z)$ be a nonconstant entire function. If $f(z) = 0 \Rightarrow f'(z) = 0$, and f and f' share b IM, then $f \equiv f'$.*

Actually, [LüXuYi, Corollary 1.1] states more generally that $f = 0 \Rightarrow f' = 0$ and $f = b \Rightarrow f' = b$ implies $f \equiv f'$ or $f = b(\frac{A}{2}e^{\frac{z}{4}} + 1)^2$ with a nonzero constant A . The additional condition $f' = b \Rightarrow f = b$ excludes the second possibility.

A more general version of Theorem C allowing $a \neq 0$ (compare [LüXuYi, Theorem 1.2]) would also admit solutions with $f \not\equiv f'$, namely $f = Ce^{\frac{b}{b-a}z} + a$ for $b \neq 0$, and $f = a(\frac{A}{2}e^{\frac{z}{4}} + 1)^2$ for $b = 0$.

The starting point of our paper is the following observation. If a meromorphic function f shares the nonzero value b with its derivative f' , then, in contrast to f sharing b with an arbitrary meromorphic function g , this imposes the additional condition that all b -points of f are simple. (This is actually used in the proofs of all theorems above.) So one might wonder about value sharing between f and f' that does not impose any conditions on the ramification of f .

In one direction, the best that we can demand is that $f = b$ implies $f' = b$ unless for ramification reasons we have $f' = 0$. In other words, every simple b -point of f is a (not necessarily simple) b -point of f' . On the other hand we can still insist on $f' = b \Rightarrow f = b$, as this does not say anything about possible multiple b -points of f .

Definition. Let f be a meromorphic function on some domain D and b a nonzero complex number. We say that f and f' **share the value b allowing ramification** if for all $z \in D$ we have

$$f(z) = b \Rightarrow f'(z) \in \{b, 0\} \quad \text{and} \quad f'(z) = b \Rightarrow f(z) = b.$$

In other words, the b -points of f' are exactly the simple b -points of f .

In some other notation this would be written as

$$\overline{E}(b, f') = \overline{E}_1(b, f).$$

Obviously it is a generalization of f and f' sharing the value b IM. But in general it is neither weaker nor stronger than the notion of f and f' sharing their simple b -points, which we have discussed a bit in [Sch].

A priori, this sharing does not impose any restrictions on the ramification of f nor of f' (beyond the general condition that outside the poles f is ramified exactly at the zeroes of f' , which already holds without the sharing).

So it might come as a surprise that, at least for entire functions, sharing b allowing ramification actually does impose *some* restriction on the ramification.

Lemma 1.1. *Let f be a nonconstant entire function and b a nonzero complex number. If f and f' share the value b allowing ramification, then b cannot be a totally ramified value of f .*

This will be part of Theorem 2.6.

For the rest of the paper we will only consider nonconstant entire functions f . If f and f' share the nonzero value b allowing ramification, then by Lemma 1.1 the extremal case $f = b \Rightarrow f' = 0$ cannot occur. The other extremal case, $f = b \Rightarrow f' = b$ simply means that f and f' share the value b in the usual sense (IM).

Several natural questions immediately come to ones mind. However, they seem to be less trivial than one might first think.

Question 1. Give an example of a nonconstant entire function f and a nonzero value b such that f and f' share b allowing ramification, but they do *not* share b in the usual sense. In other words, give an example of sharing b allowing ramification such that at least one of the b -points of f really is multiple.

Question 2. Can a nonconstant polynomial and its derivative share a nonzero value allowing ramification?

Question 3. Let a, b be two distinct nonzero complex numbers. Let $f(z)$ be a nonconstant entire function. Assume that f and its derivative f' share the value a allowing ramification and also share the value b allowing ramification. Does this imply $f \equiv f'$?

In the next section we will give some partial results towards Question 3, and also show that under different stronger conditions the answer is positive.

2. The main results

In this section we elaborate on the question how far Theorem A, and also Theorems B and C still hold if we weaken the sharing IM to sharing allowing ramification.

If in Theorem A we only weaken one of the two sharings, the answer is easy.

Theorem 2.1. *If the nonconstant entire function f and its derivative f' share the value a IM and share the value b ($\neq 0, a$) allowing ramification, then $f \equiv f'$.*

If we weaken both sharings in Theorem A, we don't know the answer, and we cannot even offer an educated guess what will happen. As a consolation we prove the following result.

Theorem 2.2. *Let a_1, a_2, a_3 be three distinct nonzero complex numbers. Let $f(z)$ be a nonconstant entire function. If f and its derivative f' share the value a_j allowing ramification for $j = 1, 2, 3$, then $f \equiv f'$.*

Remark. Actually, the conditions in Theorem 2.2 are unnecessarily strong. As the proof will show, sharing a_1 allowing ramification and $f = a_j \Rightarrow f' \in \{a_j, 0\}$ for $j = 2, 3$ already implies $f \equiv f'$.

Next we make a first step towards Theorems B and A with weakened sharing.

Proposition 2.3. *Let a and b be two nonzero distinct complex numbers, and let $f(z)$ be a nonconstant entire function. If f and f' share the value a allowing ramification, and $f'(z) = b \Rightarrow f(z) = b$, then f is a transcendental function of order at most one.*

Corollary 2.4. *Let $f(z)$ be a nonconstant entire function. If f and f' share two different nonzero values allowing ramification, then f is a transcendental function of order at most one.*

Once we know that the order of f is at most 1, one might try to prove $f \equiv f'$ by the method from [LüXuYi] (which has also successfully been applied in [Sch]). But in the current setting we couldn't get that working. However, we extract some more information from the condition that the order is at most one.

Lemma 2.5. *Let f be a transcendental entire function of order at most one, and let b be a nonzero complex number.*

- a) *If S is a finite subset of \mathbb{C} , and $f = b \Rightarrow f' \in S$, then the derivative f' takes every nonzero value infinitely often.*
- b) *If $f' = b \Rightarrow f = b$, and f takes the value a only finitely often, then $a \neq b$ and*

$$f(z) = Ce^{\frac{b}{b-a}z} + a$$

where C is a nonzero constant.

With the same method one could also prove statements about the values of f' in part b), and about the form of f in part a). We have contented ourselves with the results we need.

Theorem 2.6 *Let f be a nonconstant entire function such that f and f' share the nonzero value b allowing ramification. Then the derivative f' has no nonzero Picard value.*

In particular, b cannot be a Picard value of f' . So b cannot be a totally ramified value of f . A fortiori, b cannot be a Picard value of f . Actually, if f is transcendental, then it must take the value b infinitely often.

Now we weaken the sharing in Theorem C. We formulate the result slightly more generally.

Proposition 2.7. *Let f be a nonconstant entire function such that f and f' share the nonzero value b allowing ramification. If f has a totally ramified value a , then $a \neq b$, and the order of f is at most one.*

In the special case where a is a Picard value of f in Proposition 2.7, we can actually nail down the function, even under a weaker sharing condition for b .

Theorem 2.8. *Let f be a transcendental entire function with $f' = b \Rightarrow f = b$ for some nonzero value b . If f has a generalized Picard value a (that is, a value that is taken only finitely often by f), then $a \neq b$ and*

$$f(z) = Ce^{\frac{b}{b-a}z} + a$$

where C is a nonzero constant. In particular, if $a = 0$ we have $f \equiv f'$.

If a in Theorem 2.8 is a true Picard value, the condition that f is transcendental is of course automatic.

3. Some normality criteria

Most results in this paper are proved using normal family arguments. In this section we provide the necessary tools.

Theorem 3.1. [LiYi, Theorem 3] *Let \mathcal{F} be a family of holomorphic functions in a domain D , and let a and b be two finite complex numbers such that $b \neq a, 0$. If, for each $f \in \mathcal{F}$ and $z \in D$, $f = a \Rightarrow f' = a$, and $f' = b \Rightarrow f = b$, then \mathcal{F} is normal in D .*

Corollary 3.2. [LiYi] *Let f be an entire function and a and b two complex numbers with $b \neq a, 0$. If $f = a \Rightarrow f' = a$, and $f' = b \Rightarrow f = b$, then f has order at most one.*

For the proofs of Theorems 2.3 and 2.8 we have to generalize this.

Theorem 3.3. *Let \mathcal{F} be a family of holomorphic functions in a domain D . Let a and b be two finite complex numbers such that $b \neq a, 0$, and let S be a finite subset of \mathbb{C} . If, for each $f \in \mathcal{F}$ and $z \in D$, $f = a \Rightarrow f' \in S$, and $f' = b \Rightarrow f = b$, then \mathcal{F} is normal in D .*

Proof. Obviously the conditions imply that we can always assume $b \notin S$.

With the stronger condition $f = a \Rightarrow f' = a$ this is Theorem 3 from [LiYi], and our proof follows their proof closely. So we will be a bit sketchy and mainly emphasize the points where one has to be careful with the more general conditions.

Setting $h(z) = f(z) - a$ we have $|h'(z)| \leq M + 1$ when $h(z) = 0$ where M is the maximum of the absolute values of the elements of S . We can assume that D is the unit disk and that the family $\{f(z) - a : f \in \mathcal{F}\}$ is not normal at 0.

To bring this to a contradiction, as in [LiYi], we use a version of the Pang-Zalcman Lemma [PaZa, Lemma 2], namely the one with

$$g_n(\xi) = \rho_n^{-1} \{f_n(z_n + \rho_n \xi) - a\} \rightarrow g(\xi)$$

locally uniformly with respect to the spherical metric on \mathbb{C} , where $g(\xi)$ is a nonconstant entire function satisfying

$$g^\#(\xi) \leq g^\#(0) = M + 2.$$

First we prove $g = 0 \Rightarrow g' \in S$. If $g(\xi_0) = 0$, by Hurwitz's Theorem there exist $\xi_n \rightarrow \xi_0$ as $n \rightarrow \infty$ such that for sufficiently large n

$$g_n(\xi_n) = \rho_n^{-1} \{f_n(z_n + \rho_n \xi_n) - a\} = 0.$$

So $f_n(z_n + \rho_n \xi_n) = a$ and hence $g'_n(\xi_n) = f'_n(z_n + \rho_n \xi_n) \in S$. Since $g'_n(\xi_n) \rightarrow g'(\xi_0)$ and S is discrete, we get $g'(\xi_0) \in S$.

Next we prove that $g'(\xi) \neq b$ on \mathbb{C} . If $g'(\xi) \equiv b$, then $g(\xi) = b\xi + c$, which together with $g = 0 \Rightarrow g' \in S$ contradicts $b \notin S$. Thus $g'(\xi) \not\equiv b$, and we can argue exactly as in [LiYi] to get $g'(\xi) \neq b$.

Since $g'(\xi)$ is of order at most one, we now have $g'(\xi) = b + e^{b_0 + b_1 \xi}$ with constants b_0, b_1 .

If $b_1 \neq 0$, as in [LiYi, p.56] we get a contradiction, recalling that $g = 0 \Rightarrow g' \in S$ and $\{|s - b| : s \in S\}$ is bounded.

If $b_1 = 0$, we have $g(\xi) = (b + e^{b_0})\xi + c_0$. From $g = 0 \Rightarrow g' \in S$ we get $b + e^{b_0} \in S$, and hence the contradiction $g^\#(0) < M + 2$. \square

Corollary 3.4. *Let a and b be two finite complex numbers such that $b \neq a, 0$, and let S be a finite subset of \mathbb{C} . If f is a transcendental entire function with $f = a \Rightarrow f' \in S$, and $f' = b \Rightarrow f = b$, then f has order at most one.*

Proof. This is a standard conclusion that has been applied in countless papers. By Theorem 3.3 the family $\{f(z + \omega) : \omega \in \mathbb{C}\}$ is normal on \mathbb{C} , that is, f is a normal entire function and hence of order at most 1. See for example [Mi, p.198] and [Mi, p.211] for a proof of the last implication. \square

Remark. It seems that, with practically the same proof, Theorem 3.3 and Corollary 3.4 still hold under the much weaker condition that S is a closed, bounded set that does not contain b .

We also mention the following result which can be applied to some of the situations we are interested in.

Theorem 3.5. [Li, Theorem 2.3] *Let \mathcal{F} be a family of analytic functions in a domain D of the complex plane, a, b, c, d complex numbers with $a \neq b$, and M a positive number. Suppose that for all $f \in \mathcal{F}$*

$$(i) |f'(z)| \leq M \text{ for } z \in (f - a)^{-1}(0) \cup (f - b)^{-1}(0); \text{ and}$$

$$(ii) (f' - c)^{-1}(0) \subseteq (f - d)^{-1}(0).$$

Then \mathcal{F} is a normal family in D .

We also need a result that apparently goes back at least to Milloux (see [Ha, page 9]).

Theorem 3.6. *Let f be a transcendental entire function. If f' takes the nonzero value b only finitely often, then f must take every value infinitely often.*

4. Proofs of the main results

The first one is easy.

Proof of Theorem 2.1. If $a = 0$, then $f' = 0 \Rightarrow f = 0$ implies that b is actually also shared IM; so we are in the situation of Theorem A.

If $a \neq 0$, Theorem 2.1 is just a special case of the more general Theorem B. \square

The main work for Theorem 2.2 is in the following recent result.

Theorem 4.1. [Sch, Theorem 2.4] *Let a_1, a_2, a_3 be three distinct complex numbers.*

Nonconstant entire functions f with $f \not\equiv f'$ and

$$f = a_j \Rightarrow f' \in \{a_j, 0\}$$

for $j = 1, 2, 3$ exist if and only if $a_j = \zeta^j a_3$ with ζ being a third root of unity, that is, if $(X - a_1)(X - a_2)(X - a_3)$ is of the form $X^3 - \delta$.

Moreover, functions with this property necessarily are of the form

$$f(z) = \frac{4\delta}{27\beta^2} e^{\frac{2}{3}z} + \beta e^{-\frac{1}{3}z}$$

with a nonzero constant β .

Conversely, every function of this form has the stronger property that every simple a_j -point of f is a simple a_j -point of f' for $j = 1, 2, 3$.

Remark. Note that there is an unfortunate typo in [Sch] in the formulation of Theorem 2.4 as well as towards the end of its proof; the term $e^{\frac{2}{3}z}$ is on both occasions given incorrectly as $e^{\frac{3}{2}z}$.

Proof of Theorem 2.2. If $f \not\equiv f'$, then by Theorem 4.1 we must have $a_j = \zeta^j c$ and $f = \frac{4c^3}{27}t^2 + \frac{1}{t}$ where $t = \frac{1}{\beta}e^{\frac{1}{3}z}$. Correspondingly, $f' = \frac{8c^3}{81}t^2 - \frac{1}{3t}$. In particular, if $t = \frac{3(2+\sqrt{6})}{4c}$, then $f' = c$, one of the three shared values, but $f = (\sqrt{6} - \frac{1}{2})c \neq c$. \square

For the proof of Proposition 2.3 we need the following auxiliary result.

Lemma 4.2. *Let a, b be two distinct complex numbers. The problem*

$$f' = a \Rightarrow f = a \quad \text{and} \quad f' = b \Rightarrow f = b \tag{*}$$

has no polynomial solutions of degree bigger than 2.

Proof. If f is entire and $(*)$ holds, the auxiliary function

$$h = \frac{(f - f')f''}{(f' - a)(f' - b)}$$

is entire. If moreover f is a polynomial of degree at least 3, then for degree reasons h is a nonzero constant. After scaling everything we can assume $f = z^n + cz^{n-1} + \dots$ with $n \geq 3$. Calculating the two highest terms of the numerator and denominator of h , we obtain the contradiction that

$$\frac{n(n-1)z^{2n-2} + (n-1)(2(n-1)c - n^2)z^{2n-3} + \dots}{n^2z^{2n-2} + 2n(n-1)cz^{2n-3} + \dots}$$

is constant. \square

Remark. For $a + b \neq 0$, the quadratic polynomial $f(z) = \frac{a+b}{4}z^2 + \frac{ab}{a+b}$ is a solution of (*). But it seems to be a nontrivial problem whether in the case $a + b = 0$ there are entire solutions of (*) other than the obvious ones $f \equiv f'$ or f' a constant different from a and b .

Proof of Proposition 2.3. Polynomials of degree 1 and 2 are easily excluded. So f must be transcendental by Lemma 4.2. By Corollary 3.4 the order of f is bounded by 1. \square

One can also prove Corollary 2.4 directly. For that one can manage without Corollary 3.4, as, due to the stronger conditions, one can get normality and the bound on the order from Theorem 3.5.

Proof of Lemma 2.5. a) Assume that f' takes the nonzero value a only finitely often. Since by [YaYi, Theorem 1.21] the order of f' is also at most 1, the Hadamard Factorization Theorem [YaYi, Theorem 2.5] implies

$$f'(z) = a + P(z)e^{\lambda z}$$

with $\lambda \neq 0$ and a polynomial $P(z)$. Thus

$$f(z) = B + az + Q(z)e^{\lambda z}$$

with a constant B and a polynomial $Q(z)$ with $P = Q' + \lambda Q$. By Theorem 3.6 there are infinitely many z_0 with $f(z_0) = b$. For these we must have

$$e^{\lambda z_0} = \frac{b - B - az_0}{Q(z_0)}.$$

Moreover,

$$f'(z_0) = a + \lambda b - \lambda B - \lambda az_0 + \frac{(b - B - az_0)Q'(z_0)}{Q(z_0)}$$

lies in the finite set S for every such z_0 . This is only possible if the rational function

$$-\lambda az + \frac{(b - B - az)Q'(z)}{Q(z)} = \frac{-\lambda azQ(z) + (b - B - az)Q'(z)}{Q(z)}$$

is constant, which cannot be for degree reasons.

b) If f takes the value a only finitely often, then by the Hadamard Factorization Theorem

$$f(z) = a + P(z)e^{\lambda z}$$

with $\lambda \neq 0$ and a polynomial $P(z)$. Theorem 3.6 together with $f' = b \Rightarrow f = b$ implies $a \neq b$ and that there are infinitely many z_0 with $f'(z_0) = b$. Then an argumentation similar to the one in part a) shows that P must be a constant and

$$\lambda = \frac{b}{b-a}.$$

□

Proof of Theorem 2.6. The very last claim is an immediate consequence of $f' = b \Rightarrow f = b$ and Theorem 3.6.

Now suppose that a is a nonzero Picard value of f' . We only have show that this implies that f has order at most 1. Then Lemma 2.5 furnishes the desired contradiction.

If $a \neq b$, we can to that purpose apply Corollary 3.2 (with the roles of a and b interchanged), as we vacuously have $f' = a \Rightarrow f = a$.

If $a = b$, we have $f = b \Rightarrow f' = 0$ and vacuously $f' = b \Rightarrow f = 2b$. Hence the function $h(z) = f(z) - b$ satisfies the conditions of Corollary 3.2 and thus has order at most 1. □

Proof of Proposition 2.7. By Theorem 2.6 we have $a \neq b$. So we can apply Corollary 3.4 to bound the order.

Alternatively, we could get normality and order at most 1 from Theorem 3.5. Or we could consider $h(z) = f(\frac{b-a}{b}z) - a$. Then all zeroes of h are multiple, and h and its derivative share the value $b - a$ allowing ramification. So Corollary 3.2 suffices to bound the order. □

Proof of Theorem 2.8. Theorem 3.6 together with $f' = b \Rightarrow f = b$ immediately implies $a \neq b$. Moreover, we have of course $f = a \Rightarrow f' \in S$ with a finite set S . So by Corollary 3.4 the order of f is at most 1. Then Lemma 2.5 takes care of the rest. □

As an afterthought we mention that in final instance the proofs of almost all results use one version or another of the famous Zalcman Lemma.

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